

# Combinatorial rigidity and independence of generalized pinned subspace-incidence constraint systems

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## Abstract

Given a hypergraph  $H$  with  $m$  hyperedges and a set  $X$  of  $m$  pins, i.e. globally fixed subspaces in Euclidean space  $\mathbb{R}^d$ , a *pinned subspace-incidence system* is the pair  $(H, X)$ , with the constraint that each pin in  $X$  lies on the subspace spanned by the point realizations in  $\mathbb{R}^d$  of vertices of the corresponding hyperedge of  $H$ . We are interested in combinatorial characterization of pinned subspace-incidence systems that are *minimally rigid*, i.e. those systems that are guaranteed to generically yield a locally unique realization. As is customary, this is accompanied by a characterization of generic independence as well as rigidity.

In a previous paper [13], we used pinned subspace-incidence systems towards solving the *fitted dictionary learning* problem, i.e. dictionary learning with specified underlying hypergraph, and gave a combinatorial characterization of minimal rigidity for a more restricted version of pinned subspace-incidence system, with  $H$  being a uniform hypergraph and pins in  $X$  being 1-dimension subspaces. Moreover in a recent paper [2], the special case of pinned line incidence systems was used to model biomaterials such as cellulose and collagen fibrils in cell walls. In this paper, we extend the combinatorial characterization to general pinned subspace-incidence systems, with  $H$  being a non-uniform hypergraph and pins in  $X$  being subspaces with arbitrary dimension. As there are generally many data points per subspace in a dictionary learning problem, which can only be modeled with pins of dimension larger than 1, such an extension enables application to a much larger class of fitted dictionary learning problems.

## 1 Introduction

A *pinned subspace-incidence system*  $(H, X)$  is an incidence constraint system specified as a hypergraph  $H$  together with a set  $X$  of subspaces or *pins* in  $\mathbb{R}^d$  in one-to-one correspondence with the hyperedges of  $H$ . A realization of  $(H, X)$  assigns points in  $\mathbb{R}^d$  to the vertices of  $H$ , thereby subspaces to the hyperedges of  $H$ . The subspace in  $\mathbb{R}^d$  corresponding to a hyperedge of  $H$  contains

the associated pin from  $X$ . We are interested in characterization of pinned subspace-incidence systems that are *minimally rigid*, i.e. those systems that are guaranteed to generically yield a locally unique realization.

In a previous paper [13], we used pinned subspace-incidence systems towards solving *fitted dictionary learning* problems, i.e. dictionary learning with specified underlying hypergraphs. *Dictionary learning* (aka sparse coding) is the problem of obtaining a set *dictionary vectors* that sparsely represent a set of given data points in  $\mathbb{R}^d$ . Geometrically, such a sparse representation can be viewed as a subspace arrangement spanned by the dictionary vectors that contains all the data points. In fitted dictionary learning, the underlying hypergraph  $H$  of the subspace arrangement is specified, and the problem becomes a pinned subspace-incidence systems with the pins corresponding to the span of data points on each subspace.

Moreover in a recent paper [2], we have used pinned subspace-incidence systems in modeling biomaterials such as cross-linking cellulose and collagen microfibrils in cell walls [3, 4, 14]. In such materials, each fibril is attached to some fixed larger organelle/membrane at one site, and cross-linked at two locations with other fibrils. Consequently, they can be modeled using a pinned line-incidence system with  $H$  being a graph, where each fibril is modeled as an edge of  $H$  with the two cross-linkings as its two vertices, and the attachment is modeled as the corresponding pin.

We gave in [13] a combinatorial characterization of minimal rigidity for a restricted version of pinned subspace-incidence system, with the underlying hypergraph  $H$  being a uniform hypergraph and pins in  $X$  being 1-dimension subspaces.

## 2 Contributions

In this paper, we extend the combinatorial characterization of minimal rigidity to general pinned subspace-incidence systems, where  $H$  can be any non-uniform hypergraph, and each pin in  $X$  is a subspace with arbitrary dimension. Such an extension enables application to a much larger class of fitted dictionary learning problems, since there are generally many data points per subspace in a dictionary learning problem, which can only be modeled with pins of dimension larger than 1.

As in our previous paper [13], we apply the classic method of White and Whiteley [18] to combinatorially characterize the rigidity of general pinned subspace-incidence systems. The primary technique is using the Laplace decomposition of the *rigidity matrix*, which corresponds to a *map-decomposition* [15] of the underlying hypergraph. The polynomial resulting from the Laplace decomposition is called the *pure condition*, which characterizes the conditions that the framework has to avoid for the combinatorial characterization to hold.

Previous works on related types of geometric constraint frameworks include pin-collinear body-pin frameworks [9], direction networks [20], slider-pinning rigidity [16], body-cad constraint system [8],  $k$ -frames [18, 19], and affine rigid-

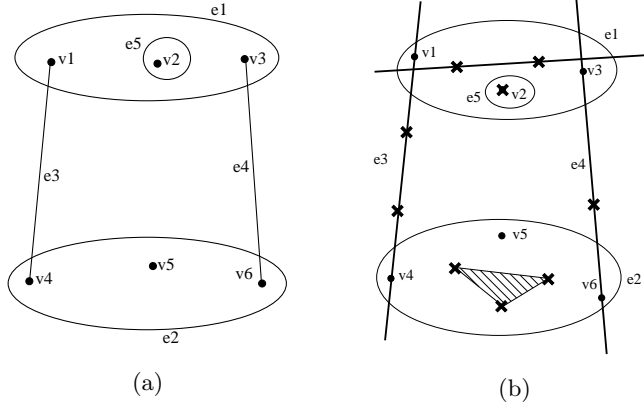


Figure 1: (a) A rank-3 non-uniform hypergraph with 6 vertices and 5 edges, where  $s_1 = s_2 = 3$ ,  $s_3 = s_4 = 2$ ,  $s_5 = 1$ . (b) A pinned subspace-incidence framework in  $d = 4$  (projectivized in  $\mathbb{P}^3(\mathbb{R})$ ) on the hypergraph from (a), with  $m_1 = m_3 = 2$ ,  $m_2 = 3$ ,  $m_4 = m_5 = 1$ , where the crosses on each hyperedge represent points spanning the associated pin.

ity [6]. However, we are not aware of any previous results on systems that are similar to pinned subspace-incidence systems.

### 3 Preliminaries

In this section, we introduce the formal definition of pinned subspace-incidence systems and basic concepts in combinatorial rigidity.

A *hypergraph*  $H = (V, E)$  is a set  $V$  of vertices and a set  $E$  of hyperedges, where each hyperedge is a subset of  $V$ . The *rank*  $r(H)$  of a hypergraph  $H$  is the maximum cardinality of any edge in  $E$ , i.e.  $r(H) = \max_{e_k \in E} s(e_k)$ , where  $s(e_k)$  denotes the cardinality of the hyperedge  $e_k$ . A hypergraph is *s-uniform* if all edges in  $E$  have the same cardinality  $s$ . A *configuration* or *realization* of a hypergraph  $H = (V, E)$  in  $\mathbb{R}^d$  is a mapping of points in  $\mathbb{R}^d$  to the vertices of  $H$ ,  $p : V \rightarrow \mathbb{R}^d$ . When there is no ambiguity, we simply use  $p_i$  to denote the point  $p(v_i)$ ,  $p(e_k)$  to denote the set of points  $\{p(v_i) | v_i \in e_k\}$ , and  $s_k$  to denote the cardinality  $s(e_k)$ .

An example of a rank-3 hypergraph is given in Figure 1a.

In the following, we use  $\langle S \rangle$  to denote the subspace *spanned* by a set  $S$  of points in  $\mathbb{R}^d$ .

**Definition 1** (Pinned Subspace-Incidence System). A *pinned subspace-incidence system* in  $\mathbb{R}^d$  is a pair  $(H, X)$ , where  $H = (V, E, m)$  is a weighted hypergraph with hyperedges of rank  $r(H) < d$ , and  $X = \{x_1, x_2, \dots, x_{|E|}\}$  is a set of *pins* (subspaces of  $\mathbb{R}^d$ ) in one-to-one correspondence with the hyperedges of  $H$ . Here the weight assignment is a function  $m : E(H) \rightarrow \mathbb{Z}^+$ , where  $m(e)$  denotes the

dimension of the pin associated with the hyperedge  $e$ . Often we ignore the weight  $m$  and just refer to the hypergraph  $(V, E)$  as  $H$ .

A *pinned subspace-incidence framework* realizing the pinned subspace-incidence system  $(H, X)$  is a triple  $(H, X, p)$ , where  $p$  is a realization of  $H$ , such that for all pins  $x_k \in X$ ,  $x_k$  is contained in  $\langle p(e_k) \rangle$ , the subspace spanned by the set of points realizing the vertices of the hyperedge  $e_i$  corresponding to  $x_k$ .

We may write  $m(x_k)$  or simply  $m_k$  in substitute of  $m(e_k)$ , where  $x_k$  is the pin associated with the hyperedge  $e_k$ .

Since we only care about incidence relations, we projectivize the Euclidean space  $\mathbb{R}^d$  to treat the pinned subspace-incidence system in the real projective space  $\mathbb{P}^{d-1}(\mathbb{R})$ , and use the same notation for the pins and hypergraph realization when the meaning is clear from the context.

Figure 1b gives an example of a pinned subspace-incidence framework in the projective space  $\mathbb{P}^{d-1}(\mathbb{R})$  with  $d = 4$ , where each pin is the subspace spanned by the set of cross-denoted points on the corresponding hyperedge.

**Definition 2.** A pinned subspace-incidence system  $(H, X)$  is *independent* if none of the algebraic constraints is in the ideal generated by others, which generically implies the existence of a realization. It is *rigid* if there exist at most finitely many realizations. It is *minimally rigid* if it is both rigid and independent. It is *globally rigid* if there exists at most one realization.

## 4 Algebraic Representation and Linearization

In the following, we use  $A[R, C]$  to denote a submatrix of a matrix  $A$ , where  $R$  and  $C$  are respectively index sets of the rows and columns contained in the submatrix. In addition,  $A[R, \cdot]$  represents the submatrix containing row set  $R$  and all columns, and  $A[\cdot, C]$  represents the submatrix containing column set  $C$  and all rows.

### 4.1 Algebraic Representation

A pin  $x_k$  associated with a hyperedge  $e_k = \{v_1^k, v_2^k, \dots, v_{s_k}^k\}$  of cardinality  $s_k$  is constrained to be contained in the subspace  $\langle p(e_k) \rangle$  spanned by the point set  $\{p_1^k, p_2^k, \dots, p_{s_k}^k\}$ . As  $x_k$  is a subspace of dimension  $m_k - 1$  in  $\mathbb{P}^{d-1}(\mathbb{R})$ , we can pick a set of  $m_k$  points  $\{x_1^k, x_2^k, \dots, x_{m_k}^k\}$  spanning  $x_k$  from  $x_k$ . Now the constraint is equivalent to requiring each such point  $x_l^k$  to lie on  $\langle p(e) \rangle$ , for  $1 \leq l \leq m_k$ . We call each such point  $x_l^k$  a *multipin* as it acts like a pin with  $m(x_l^k) = 1$ .

Using homogeneous coordinates  $p_i^k = [p_{i,1}^k \ p_{i,2}^k \ \dots \ p_{i,d-1}^k]$  and  $x_l^k = [x_{l,1}^k \ x_{l,2}^k \ \dots \ x_{l,d-1}^k]$ , we write this incidence constraint for each point  $x_l$

by letting all the  $s_k \times s_k$  minors of the  $s_k \times (d-1)$  matrix

$$E_l^k = \begin{bmatrix} p_1^k - x_l^k \\ p_2^k - x_l^k \\ \vdots \\ p_{s_k}^k - x_l^k \end{bmatrix}$$

be zero. There are  $\binom{d-1}{s_k}$  minors, giving  $\binom{d-1}{s_k}$  equations. Note that any  $d - s_k$  of these  $\binom{d-1}{s_k}$  equations are independent and span the rest. So we can write the incidence constraint as  $(d - s_k)$  independent equations:

$$\det(E_l^k[\cdot, C(t)]) = 0, \quad 1 \leq t \leq d - s_k \quad (1)$$

where  $C(t)$  denote the following index sets of columns in  $E$ :

$$C(t) = \{1, 2, \dots, s_k - 1\} \cup \{s_k - 1 + t\} \quad 1 \leq t \leq d - s_k$$

In other words,  $C(t)$  contains the first  $s_k - 1$  columns together with Column  $s_k - 1 + t$ .

Now the incidence constraint for the pin  $x_k$  is represented as  $m_k(d - s_k)$  equations for all the  $m_k$  multipins  $\{x_1^k, x_2^k, \dots, x_{m_k}^k\}$ . Consequently, the pinned subspace-incidence problem reduces to solving a system of  $\sum_{k=1}^{|E|} m_k(d - s_k)$  equations, each of form (1). We denote this algebraic system by  $(H, X)(p) = 0$ .

## 4.2 Genericity

We are interested in characterizing *minimal rigidity* of pinned subspace-incidence systems. However, checking independence relative to the ideal generated by the variety is computationally hard and best known algorithms, such as computing Gröbner basis, are exponential in time and space [11]. However, the algebraic system can be linearized at *generic* or *regular* (non-singular) points, whereby the independence and rigidity of the algebraic system  $(H, X)(p) = 0$  reduces to linear independence and maximal rank at *generic* frameworks.

In algebraic geometry, a property being generic intuitively means that the property holds on the open dense complement of an (real) algebraic variety. Formally,

**Definition 3.** A pinned subspace-incidence system  $(H, X)$  is *generic* w.r.t. a property  $Q$  if and only if there exists a neighborhood  $N(X)$  such that for all systems  $(H, X')$  with  $X' \in N(X)$ ,  $(H, X')$  satisfies  $Q$  if and only if  $(H, X)$  satisfies  $Q$ .

Similarly, a framework  $(H, X, p)$  is *generic* w.r.t. a property  $Q$  if and only if there exists a neighborhood  $N(p)$  such that for all frameworks  $(H, X, q)$  with  $q \in N(p)$ ,  $(H, X, q)$  satisfies  $Q$  if and only if  $(H, X, p)$  satisfies  $Q$ .

Furthermore we can define generic properties in terms of the underlying weighted hypergraph.

**Definition 4.** A property  $Q$  of pinned subspace-incidence systems is *generic* (i.e, becomes a property of the underlying weighted hypergraph alone) if for any weighted hypergraph  $H = (V, E, m)$ , either all generic (w.r.t.  $Q$ ) systems  $(H, X)$  satisfies  $Q$ , or all generic (w.r.t.  $Q$ ) systems  $(H, X)$  do not satisfy  $Q$ .

Once an appropriate notion of genericity is defined, we can treat  $Q$  as a property of a hypergraph. The primary activity of the area of combinatorial rigidity is to give purely combinatorial characterizations of such generic properties  $Q$ . In the process of drawing such combinatorial characterizations, the notion of genericity may have to be further restricted by so-called pure conditions that are necessary for the combinatorial characterization to go through (we will see this in the proof of Theorem 11).

### 4.3 Linearization as Rigidity Matrix

Next we follow the approach taken by traditional combinatorial rigidity theory [1, 7] to show that rigidity and independence (based on nonlinear polynomials) of pinned subspace-incidence systems are generically properties of the underlying weighted hypergraph  $H$ , and can furthermore be captured by linear conditions in an infinitesimal setting. Specifically, Lemma 6 shows that rigidity of a pinned subspace-incidence system is equivalent to existence of a full rank *rigidity matrix*, obtained by taking the Jacobian of the algebraic system  $(H, X)(p)$  at a regular point.

A *rigidity matrix* of a framework  $(H, X, p)$  is a matrix whose kernel is the infinitesimal motions (flexes) of  $(H, X, p)$ . A framework is *infinitesimally rigid* if the space of infinitesimal motion is trivial, i.e. the rigidity matrix has full rank. To define a rigidity matrix for a pinned subspace-incidence framework  $(H, X, p)$ , we take the Jacobian of the algebraic system  $(H, X)(p) = 0$  by taking partial derivatives with respect to the coordinates of  $p_i$ 's. In the Jacobian, each vertex  $v_i$  has  $d - 1$  corresponding columns, and each pin  $x_k$  associated with the hyperedge  $e_k = \{v_1^k, v_2^k, \dots, v_{s_k}^k\}$  has  $m_k(d - s_k)$  corresponding rows, where each equation  $\det(E_l^k[\cdot, C(t)]) = 0$  (1), i.e. Equation  $t$  of the multipin  $x_l^k$ , gives the following row (the columns corresponding to vertices not in  $e_k$  are all zero):

$$\begin{aligned} & \left[ 0, \dots, 0, 0, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{1,1}^k}, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{1,2}^k}, \dots, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{1,d-1}^k}, 0, 0, \right. \\ & \quad \dots, 0, 0, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{2,1}^k}, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{2,2}^k}, \dots, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{2,d-1}^k}, 0, 0, \dots \\ & \quad \dots \dots \dots \\ & \quad \left. \dots, 0, 0, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{s_k,1}^k}, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{s_k,2}^k}, \dots, \frac{\partial \det(E_l^k[\cdot, C(t)])}{\partial p_{s_k,d-1}^k}, 0, \dots, 0 \right] \end{aligned} \quad (2)$$

Let  $V^k$  be the matrix whose rows are coordinates of  $p_1^k, p_2^k, \dots, p_{s_k}^k$ :

$$\begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,d-1} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s_k,1} & p_{s_k,2} & \cdots & p_{s_k,d-1} \end{bmatrix}$$

Let  $V_t^k$  be the  $V^k[\cdot, C(t)]$ , i.e. the  $s_k \times s_k$  submatrix of  $V^k$  containing only columns in  $C(t)$ . Let  $V_{t,j}^k$  be the matrix obtained from  $V_t^k$  by replacing the column corresponding to Coordinate  $j$  with the all-ones vector  $(1, 1, \dots, 1)$  for  $j \in C(t)$ , and the zero matrix for  $j \notin C(t)$ . Let  $D_{t,j}^k$  be the determinant of  $V_{t,j}^k$ . Let  $x_l = \sum_{i=1}^{s_k} b_i^{k,l} v_i^k$  (note that  $\sum_{i=1}^{s_k} b_i^{k,l} = 1$ ). Now (2) can be rewritten in the following simplified form:

$$\begin{aligned} r_{t,l}^k = & \left[ 0, \dots, 0, 0, D_{t,1}^k b_1^{k,l}, D_{t,2}^k b_2^{k,l}, \dots, D_{t,d-1}^k b_{s_k}^{k,l}, 0, 0, \right. \\ & \dots, 0, 0, D_{t,1}^k b_2^{k,l}, D_{t,2}^k b_2^{k,l}, \dots, D_{t,d-1}^k b_2^{k,l}, 0, 0, \dots \\ & \dots \dots \dots \\ & \left. \dots, 0, 0, D_{t,1}^k b_{s_k}^{k,l}, D_{t,2}^k b_{s_k}^{k,l}, \dots, D_{t,d-1}^k b_{s_k}^{k,l}, 0, 0, \dots, 0 \right] \quad (3) \end{aligned}$$

Each vertex  $v_i^k$  has the entries  $D_{t,j}^k b_i^{k,l}$ ,  $1 \leq j \leq d-1$  in its  $d-1$  columns, among which exactly  $s_k$  entries with  $j \in C(t)$ , i.e. the first  $s_k - 1$  columns together with Column  $s_k - 1 + t$ , are generically non-zero. Note that the terms  $D_{t,s_k-1+t}^k$  are equal for all  $t$ , so we may just use  $D^k$  to denote it.

For each  $1 \leq t \leq d - s_k$ , there are  $m_k$  rows as (3), where each multipin  $x_l^k$  corresponds to the row  $r_{t,l}^k$  for  $1 \leq l \leq m_k$ . These  $m_k$  rows have exactly the same row pattern except for different  $b_i^{k,l}$ s:

$$\begin{bmatrix} \dots & v_{1,1} & v_{1,2} & \dots & v_{1,d-1} & \dots & v_{s_k,1} & v_{s_k,2} & \dots & v_{s_k,d-1} & \dots \\ \dots & D_{t,1}^k b_1^{k,1} & D_{t,2}^k b_1^{k,1} & \dots & D_{t,d-1}^k b_1^{k,1} & \dots & D_{t,1}^k b_{s_k}^{k,1} & D_{t,2}^k b_{s_k}^{k,1} & \dots & D_{t,d-1}^k b_{s_k}^{k,1} & \dots \\ \dots & D_{t,1}^k b_1^{k,2} & D_{t,2}^k b_1^{k,2} & \dots & D_{t,d-1}^k b_1^{k,2} & \dots & D_{t,1}^k b_{s_k}^{k,2} & D_{t,2}^k b_{s_k}^{k,2} & \dots & D_{t,d-1}^k b_{s_k}^{k,2} & \dots \\ & & & & & \ddots & & & & & \\ \dots & D_{t,1}^k b_1^{k,m_k} & D_{t,2}^k b_1^{k,m_k} & \dots & D_{t,d-1}^k b_1^{k,m_k} & \dots & D_{t,1}^k b_{s_k}^{k,m_k} & D_{t,2}^k b_{s_k}^{k,m_k} & \dots & D_{t,d-1}^k b_{s_k}^{k,m_k} & \dots \end{bmatrix}$$

**Example 1.** For  $d = 4$ , consider a pin  $x$  with  $m(x) = 2$  associated with the hyperedge  $e = \{v_1, v_2\}$ . The pin has the following  $m(x) \cdot (d - s(e)) = 4$  rows in the simplified Jacobian (the index  $k$  is omitted):

$$\begin{aligned} t=1, l=1 & \quad \begin{bmatrix} \dots & v_{1,1} & v_{1,2} & v_{1,3} & \dots & v_{2,1} & v_{2,2} & v_{2,3} \\ \dots & D_{1,1} b_1^1 & D b_1^1 & 0 & \dots & D_{1,1} b_2^1 & D b_2^1 & 0 \\ \dots & D_{1,1} b_1^2 & D b_1^2 & 0 & \dots & D_{1,1} b_2^2 & D b_2^2 & 0 \end{bmatrix} \\ t=1, l=2 & \quad \begin{bmatrix} \dots & D_{2,1} b_1^1 & 0 & D b_1^1 & \dots & D_{2,1} b_2^1 & 0 & D b_2^1 \\ \dots & D_{2,1} b_1^2 & 0 & D b_1^2 & \dots & D_{2,1} b_2^2 & 0 & D b_2^2 \end{bmatrix} \end{aligned}$$

We define the *rigidity matrix*  $M(H, X, p)$  or simply  $M(p)$  for a pinned subspace-incidence framework  $(H, X, p)$  to be the simplified Jacobian matrix obtained above, where each row has form (3). It is a matrix of size  $\sum_k m_k(d - s_k)$  by  $n(d - 1)$ .

**Definition 5.** A pinned subspace-incidence framework  $(H, X, p)$  and the corresponding system  $(H, X)$  is *generic* if  $p$  and  $X$  are regular/non-singular points with respect to the algebraic system  $(H, X)(p) = 0$ .

We use  $M(H)$  or simply  $M$  to denote the generic rigidity matrix for a weighted hypergraph  $H$ . Note that the rank of  $M$  cannot be less than the rank of  $M(p)$  for any specific realization  $p$ .

**Lemma 6.** *Generic infinitesimal rigidity of a pinned subspace-incidence framework  $(H, X, p)$  is equivalent to generic rigidity of the system  $(H, X)$ .*

The proof of Lemma 6 follows the traditional combinatorial rigidity approach [1] and is given in the Appendix.

## 5 Combinatorial rigidity characterization

### 5.1 Required hypergraph properties

This section introduces pure hypergraph properties and definitions that will be used in stating and proving our main theorem.

**Definition 7.** A hypergraph  $H = (V, E)$  is  $(k, 0)$ -sparse if for any  $V' \subset V$ , the induced subgraph  $H' = (V', E')$  satisfies  $|E'| \leq k|V'|$ . A hypergraph  $H$  is  $(k, 0)$ -tight if  $H$  is  $(k, 0)$ -sparse and  $|E| = k|V|$ .

This is a special case of the  $(k, l)$ -sparsity condition that was formally studied widely in the geometric constraint solving and combinatorial rigidity literature before it was given a name in [10]. A relevant concept from graph matroids is *map-graph*, defined as follows.

**Definition 8.** An *orientation* of a hypergraph is given by identifying as the *tail* of each edge one of its endpoints. The *out-degree* of a vertex is the number of edges which identify it as the tail and connect  $v$  to  $V - v$ . A *map-graph* is a hypergraph that admits an orientation such that the out degree of every vertex is exactly one.

The following lemma from [15] follows Tutte-Nash Williams [17, 12] to give a useful characterization of  $(k, 0)$ -tight graphs in terms of maps.

**Lemma 9.** *A hypergraph  $H$  is composed of  $k$  edge-disjoint map-graphs if and only if  $H$  is  $(k, 0)$ -tight.*

Our characterization of rigidity of a weighted hypergraph  $H$  is based on map-decomposition of a *multi-hypergraph*  $\hat{H}$  obtained from  $H$ .



**Definition 10.** Given a weighted hypergraph  $H = (V, E, m)$ , the associated *multi-hypergraph*  $\hat{H} = (V, \hat{E})$  is obtained by replacing each hyperedge  $e_k$  in  $E$  with a set  $E^k$  of  $m_k(d - s_k)$  copies of *multi-hyperedges*.

A *labeling* of a multi-hypergraph  $\hat{H}$  gives a one-to-one correspondence between  $E^k$  and the set  $R^k$  of  $m_k(d - s_k)$  rows for the hyperedge  $e_k$  in the rigidity matrix  $M$ , where the multi-hyperedge corresponding to the row  $r_{t,l}^k$  is labeled  $e_{t,l}^k$ .

## 5.2 Characterizing rigidity

In this section, we apply [18] to give combinatorial characterization for minimal rigidity of pinned subspace-incidence systems.

**Theorem 11 (Main Theorem).** *A pinned subspace-incidence system is generically minimally rigid if and only if:*

1. *The underlying weighted hypergraph  $H = (V, E, m)$  satisfies  $\sum_{k=1}^{|E|} m_k(d - s_k) = (d - 1)|V|$ , and  $\sum_{e_k \in E'} m_k(d - s_k) \leq (d - 1)|V'|$  for every vertex induced subgraph  $H' = (V', E')$ . In other words, the associated multi-hypergraph  $\hat{H} = (V, \hat{E})$  has a decomposition into  $(d - 1)$  maps.*
2. *There exists a labeling of  $\hat{H}$  compatible with the map-decomposition (defined later) such that in each set  $E^k$  of multi-hyperedges,*
  - (a) *two multi-hyperedges  $e_{t_1, l_1}^k$  and  $e_{t_2, l_2}^k$  with  $l_1 = l_2$  are not contained in the same map in the map-decomposition,*
  - (b) *two multi-hyperedges  $e_{t_1, l_1}^k$  and  $e_{t_2, l_2}^k$  with  $t_1 = t_2$  do not have the same vertex as tail in the map-decomposition.*

To prove Theorem 11, we apply Laplace expansion to the determinant of the rigidity matrix  $M$ , which corresponds to decomposing the  $(d - 1, 0)$ -tight multi-hypergraph  $\hat{H}$  as a union of  $d - 1$  maps. We then prove  $\det(M)$  is not identically zero by showing that the minors corresponding to each map are not identically zero, as long as a certain polynomial called *pure condition* is avoided by the framework. The pure condition characterizes the non-genericity that the framework has to avoid in order for the combinatorial characterization to go through: see Example 4.

A Laplace expansion rewrites the determinant of the rigidity matrix  $M$  as a sum of products of determinants (brackets) representing each of the coordinates taken separately. In order to see the relationship between the Laplace expansion and the map-decomposition, we first group the columns of  $M$  into  $d - 1$  column groups  $C_j$  according to the coordinates, where columns for the first coordinate of each vertex belong to  $C_1$ , columns for the second coordinate of each vertex belong to  $C_2$ , etc.

**Example 2.** For  $d = 4$ , consider a pin  $x$  with  $m(x) = 2$  associated with the hyperedge  $e = v_1, v_2$ . The regrouped rigidity matrix has  $d - 1 = 3$  column

groups, where the pin  $x$  has the following 4 rows (the index  $k$  is omitted):

$$\begin{array}{l}
t = 1, l = 1 \\
t = 1, l = 2 \\
t = 2, l = 1 \\
t = 2, l = 2
\end{array}
\left[ \begin{array}{cccc|cccc|cccc}
& v_{1,1} & & v_{2,1} & & v_{1,2} & & v_{2,2} & & v_{1,3} & & v_{2,3} \\
\ldots & D_{1,1}b_1^1 & \ldots & D_{1,1}b_2^1 & \ldots & \ldots & Db_1^1 & \ldots & Db_2^1 & \ldots & & \\
\ldots & D_{1,1}b_1^2 & \ldots & D_{1,1}b_2^2 & \ldots & \ldots & Db_1^2 & \ldots & Db_2^2 & \ldots & & \\
\ldots & D_{2,1}b_1^1 & \ldots & D_{2,1}b_2^1 & \ldots & & & & & \ldots & Db_1^1 & \ldots & Db_2^1 & \ldots \\
\ldots & D_{2,1}b_1^2 & \ldots & D_{2,1}b_2^2 & \ldots & & & & & \ldots & Db_1^2 & \ldots & Db_2^2 & \ldots
\end{array} \right]$$

We have the following observation on the pattern of the regrouped rigidity matrix.

**Observation 1.** *In the rigidity matrix  $M$  with columns grouped into column groups, a hyperedge  $e_k$  has  $m_k(d - s_k)$  rows, each associated with a multi-hyperedge of  $e^k$  in  $\hat{H}$ . In a column group  $j$  where  $j \leq s_k - 1$ , each row associated with  $e_k$  contains  $s_k$  nonzero entries at the columns corresponding to vertices of  $e_k$ . In a Column group  $j$  where  $j = s_k - 1 + t \geq s_k$ , a row associated with a multi-hyperedge  $e_{r,l}^k$  of  $e^k$  is all zero if  $r \neq t$ ; the remaining  $m_k$  rows contains  $s_k$  nonzero entries at the columns corresponding to vertices of  $e_k$ .*

A labeling of  $\hat{H}$  compatible with a given map-decomposition can be obtained as following. We start from the last column group of  $M$  and associate each column group  $j$  with a map in the map-decomposition. For each multi-hyperedge of the map that is a copy of the hyperedge  $e^k$ , we pick a row  $r_{t,j}^k$  that is not all zero in Column groups  $j$  and label the multi-hyperedge as  $e_{t,j}^k$ . By Observation 1, this is always possible if each map contains at most  $m_k$  multi-hyperedges of the same hyperedge  $e_k$ , which must be true if there exists any labeling of  $\hat{H}$  satisfying Condition 2(a) of Theorem 11.

In the Laplace expansion

$$\det(M) = \sum_{\sigma} \left( \pm \prod_j \det M[R_j^{\sigma}, C_j] \right) \quad (4)$$

the sum is taken over all partitions  $\sigma$  of the rows into  $d - 1$  subsets  $R_1^{\sigma}, R_2^{\sigma}, \dots, R_j^{\sigma}, \dots, R_{d-1}^{\sigma}$ , each of size  $|V|$ . In other words, each summation term of (4) contains  $|V|$  rows  $R_j^{\sigma}$  from each column group  $C_j$ . Observe that for any submatrix  $M[R_j^{\sigma}, C_j]$ , each row has a common coefficient  $D_{t,j}^k$ , so

$$\det(M[R_j^{\sigma}, C_j]) = \left( \prod_{r_{t,j}^k \in R_j^{\sigma}} D_{t,j}^k \right) \det(M'[R_j^{\sigma}, C_j])$$

where each row of  $M'[R_j^{\sigma}, C_j]$  is either all zero, or of the pattern

$$[0, \dots, 0, b_1^{k,l}, b_2^{k,l}, 0, \dots, b_{s_k}^{k,l}, 0, \dots, 0] \quad (5)$$

with non-zero entries only at the  $s_k$  indices corresponding to  $v_i^k \in e_k$ .

For a fixed  $\sigma$ , we refer to a submatrix  $M[R_j^{\sigma}, C_j]$  simply as  $M_j$ .

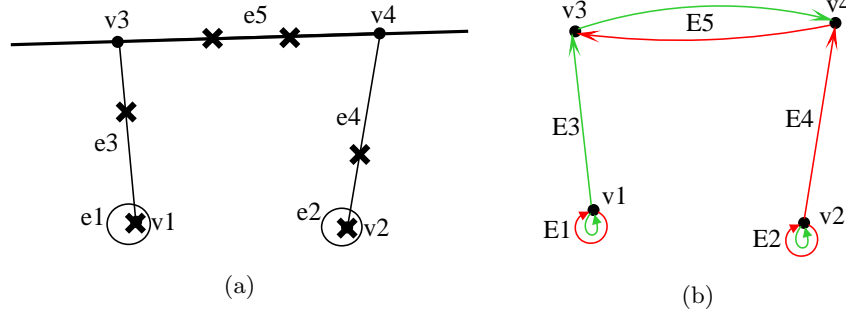


Figure 2: (a) A minimally rigid pinned subspace-incidence system in  $d = 3$ . (b) A map-decomposition of the multi-hypergraph of the system in (a), where multi-hyperedges with different colors are in different maps, and the tail vertex of each multi-hyperedge is pointed to by an arrow.

**Example 3.** Figure 2a shows a pinned subspace-incidence system in  $d = 3$  with 4 vertices and 5 hyperedges, where  $e_1 = \{v_1\}$ ,  $e_2 = \{v_2\}$ ,  $e_3 = \{v_1, v_3\}$ ,  $e_4 = \{v_2, v_4\}$ ,  $e_5 = \{v_3, v_4\}$ , and  $m_k = 1$  for all  $1 \leq k \leq 5$  except that  $m_5 = 2$ . Figure 2b gives a map-decomposition of the multi-hypergraph  $\hat{H}$  of (a). The labeling of multi-hyperedges is given in the regrouped rigidity matrix (6), where the colored rows inside the column groups constitute the submatrices  $M_j^\sigma$  in the summation term of the Laplace decomposition corresponding to the map-decomposition. The system is generically minimally rigid, as the map-decomposition and labeling of  $\hat{H}$  satisfies the conditions of Theorem 11.

$$\begin{array}{c}
 \begin{array}{cccc}
 & v_{1,1} & v_{2,1} & v_{3,1} & v_{4,1} \\
 e_{1,1}^1 & \text{D}^1 \mathbf{b}^1 & & & \\
 e_{2,1}^1 & & & & \\
 e_{1,1}^2 & & \text{D}^2 \mathbf{b}^2 & & \\
 e_{2,1}^2 & & & & \\
 e_{1,1}^3 & \text{D}_1^3 b_1^3 & & \text{D}_1^3 b_2^3 & \\
 e_{1,1}^4 & & D_1^4 b_1^4 & & D_1^4 b_2^4 \\
 e_{1,1}^5 & & & D_1^5 b_1^{5,1} & \text{D}_1^5 b_2^{5,1} \\
 e_{1,2}^5 & & & D_1^5 b_1^{5,2} & D_1^5 b_2^{5,2}
 \end{array}
 \left|
 \begin{array}{cccc}
 & v_{1,2} & v_{2,2} & v_{3,2} & v_{4,2} \\
 & \text{D}^1 \mathbf{b}^1 & & & \\
 & & \text{D}^2 \mathbf{b}^2 & & \\
 & D^3 b_1^3 & & D^3 b_2^3 & \\
 & & D^4 b_1^4 & & \text{D}^4 \mathbf{b}_2^4 \\
 & & & D_2^5 b_1^{5,1} & D_2^5 b_2^{5,1} \\
 & & & \text{D}_2^5 \mathbf{b}_1^{5,2} & D_2^5 b_2^{5,2}
 \end{array}
 \right.
 \end{array} \quad (6)$$

*Main Theorem.* First we show the only if direction. For a generically minimally rigid pinned subspace-incidence framework, the rigidity matrix  $M$  is generically full rank, so there exists at least one summation term  $\sigma$  in (4) where each

submatrix  $M_j$  is generically full rank. As the submatrices don't have any overlapping rows with each other, we can perform row elimination on  $M$  to obtain a matrix  $N$  with the same rank, where all submatrices  $M_j$  are simultaneously converted to a *permuted reduced row echelon form*  $N_j$ , where each row in  $N_j$  has exactly one non-zero entry  $\beta_i^j$  at a unique Column  $i$ . In other words, all  $N_j$ 's can be converted simultaneously to reduced row echelon form by multiplying a permutation matrix on the left of  $N$ . Now we can obtain a map-decomposition of  $\hat{H}$  by letting each map  $j$  contain multi-hyperedges corresponding to rows of the submatrix  $N_j$ , and assigning each multi-hyperedge in map  $j$  the vertex  $i$  corresponding to the non-zero entry  $\beta_i^j$  in the associated row in  $N_j$  as tail. In addition, such a map-decomposition must satisfy Condition 2 in Theorem 11:

Condition 2(a): assume two multi-hyperedges  $e_{t_1,l}^k$  and  $e_{t_2,l}^k$  are in the same map  $j$ , i.e. the rows corresponding to these two edges are included in the same submatrix  $M_j$ . If  $j > s - 1$ , one of these rows must be all-zero in  $M_j$  by Observation 1, contradicting the condition that  $M_j$  is full rank. If  $j \leq s - 1$ , both of these rows in  $M_j$  will be a multiple of the same row vector (5), contradicting the condition that  $M_j$  is full rank.

Condition 2(b): note that the rows in  $M$  corresponding to multi-hyperedges  $e_{t,l_1}^k$  and  $e_{t,l_2}^k$  have the exactly the same pattern except for different values of  $b$ 's. If  $e_{t,l_1}^k$  has vertex  $i$  as tail, after the row elimination, the column containing  $b_i^{k,l_1}$  will become the only non-zero entry of column  $i$  for  $N_{j_1}$ , while the column containing  $b_i^{k,l_2}$  will become zero in all column groups, thus  $i$  cannot be assigned as tail for  $e_{t,l_2}^k$ .

Next we show the if direction, that the conditions of Theorem 11 imply infinitesimal rigidity.

Given labeled multi-hypergraph  $\hat{H}$  with a map-decomposition satisfying the conditions of Theorem 11, we can obtain summation term  $\sigma$  in the Laplace decomposition (4) according to the labeling of  $\hat{H}$ , where each submatrix  $M_j$  contain all rows corresponding to the map associated with Column group  $j$ .

We first show that each submatrix  $M_j$  is generically full rank. According to the definition of a map-graph, the function  $\tau : \hat{E} \rightarrow V$  assigning a tail vertex to each multi-hyperedge is a one-to-one correspondence. We perform symbolic row elimination of the matrix  $M$  to simultaneously convert each  $M_j$  to its permuted reduced row echelon form  $N_j$ , where for each row of  $N_j$ , all entries are zero except for the entry  $\beta_{t,l}^k$  corresponding to the vertex  $\tau(e_{t,l}^k)$ , which is a polynomial in  $b_i^{k,l}$ 's in the submatrix  $M_j$ . Since  $M_j$  cannot contain two rows with the same  $k$  and  $l$  by Condition 2(a), the  $b_i^{k,l}$ 's in different rows of a same map are independent of each other,  $\beta_{t,l}^k \neq 0$  under a generic specialization of  $b_i^{k,l}$ . Since each row of  $N_j$  has exactly one nonzero entry and the nonzero entries from different rows are on different columns, the  $|V| \times |V|$  matrix  $N_j$  is clearly full rank. Thus  $M_j$  must also be generically full rank.

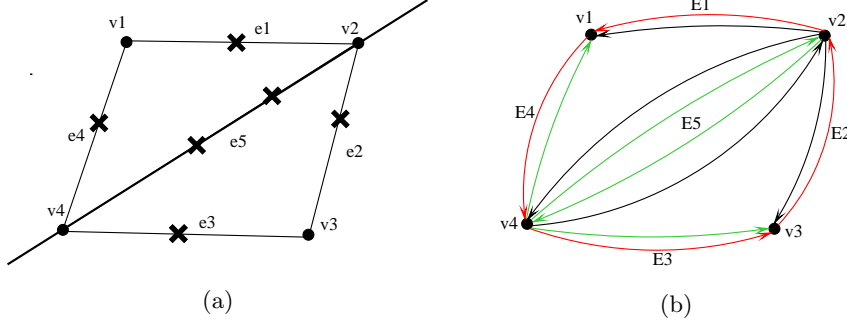


Figure 3: (a) A pinned subspace-incidence system in  $d = 4$ . (b) A map-decomposition of the multi-hypergraph of the system in (a), where multi-hyperedges with different colors are in different maps, and the tail vertex of each multi-hyperedge is pointed to by an arrow.

We conclude that

$$\det(M) = \sum_{\sigma} \left( \pm \prod_j \left( \left( \prod_{r_{t,j}^k \in R_j^{\sigma}} D_{t,j}^k \right) \det M'[R_j^{\sigma}, C_j] \right) \right) \quad (7)$$

where the sum is taken over all  $\sigma$  corresponding to a map-decomposition of  $\hat{H}$ . Generically, the summation terms of the sum (7) do not cancel with each other, since  $\det(M'[R_j^{\sigma}, C_j])$  are independent of the multi-linear coefficients  $\prod_{r_{t,j}^k \in R_j^{\sigma}} D_{t,j}^k$ , and any two rows of  $M$  are independent by Condition 2(b). This implies that  $\hat{M}$  is generically full rank.

The polynomial (7) gives the pure condition for genericity. In particular, when there is a subgraph  $(V', E')$  with  $|V'| < d$  and  $\sum_{e_k \in E'} m_k > |V'|$ , the pure condition vanishes and the system won't be minimally rigid: see Example 4.  $\square$

### 5.2.1 Pure condition

The pure condition (7) obtained in the proof of Theorem 11 characterizes the badly behaved cases that break the combinatorial characterization of infinitesimal rigidity. However, the geometric meaning of the pure condition is not completely clear. One particular condition not captured by Theorem 11 but enforced by the pure condition is that there cannot exist a subgraph  $(V', E')$  of  $H$  with  $|V'| < d$  such that  $\sum_{e_k \in E'} m_k > |V'|$ , otherwise simple counterexamples can be constructed to the characterization of the main theorem. An immediate consequence is that for any hyperedge  $e^k$ , the dimension  $m_k$  of its associated pin must be less than or equal to its cardinality  $s_k$ .

**Example 4.** Figure 3a shows a pinned subspace-incidence system in  $d = 4$  with 4 vertices and 5 hyperedges, where  $m_k$  is 2 for  $k = 5$  and is 1 otherwise. A map-

decomposition of the multi-hypergraph  $\hat{H}$  of the system is given in Figure 3b, and we can easily find a labeling of  $\hat{H}$  satisfying conditions in Theorem 11. However, the system is not minimally rigid, as generically the pin  $x_1$  will not fall on the plane spanned by pins  $x_4$  and  $x_5$ . Note that the sub-hypergraph  $(V', E')$  spanned by vertices  $v_1, v_2, v_4$  violates the pure condition as  $\sum_{e_k \in E'} m_k = m_1 + m_4 + m_5 = 4 > |V'| = 3$ .

## 6 Conclusion

In this paper, we studied the pinned subspace-incidence, a class of incidence geometric constraint system with applications in dictionary learning and bio-material modeling. We extend our results in [13] and obtain a combinatorial characterization of minimal rigidity for general pinned subspace-incidence systems with non-uniform underlying hypergraphs and pins being subspaces with arbitrary dimensions.

As future work, we plan to extend the underlying group of the pinned subspace-constraint system, i.e. consider two frameworks to be congruent if the point realization and pin set of one can be obtained from the other under the action of a certain group, for example the projective group. Another possible direction is to apply Cayley factorization [5] to find geometric interpretations of the pure conditions.

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## A Proof of Lemma 6

*Proof.* First we show that if a framework  $(H, X, p)$  is regular, infinitesimal rigidity implies rigidity. Consider the polynomial system  $(H, X)(p)$  of equations. The Implicit Function Theorem states that there exists a function  $g$ , such that  $p = g(X)$  on some open interval, if and only if the rigidity matrix  $M$  has full rank. Therefore, if the framework is infinitesimally rigid, the solutions to the algebraic system are isolated points (otherwise  $g$  could not be explicit). Since the algebraic system contains finitely many components, there are only finitely many such solution and each solution is a 0 dimensional point. This implies that the total number of solutions is finite, which is the definition of rigidity.

To show that generic rigidity implies generic infinitesimal rigidity, we take the contrapositive: if a generic framework is not infinitesimally rigid, we show that there is a finite flex. If  $(H, X, p)$  is not infinitesimally rigid, then the rank  $r$  of the rigidity matrix  $M$  is less than  $(d - 1)|V|$ . Let  $E^*$  be a set of edges in  $H$  such that  $|E^*| = r$  and the corresponding rows in  $M$  are all independent. In  $M[E^*, \cdot]$ , we can find  $r$  independent columns. Let  $p^*$  be the components of  $p$  corresponding to those  $r$  independent columns and  $p^{*\perp}$  be the remaining components. The  $r$ -by- $r$  submatrix  $M[E^*, p^*]$ , made up of the corresponding independent rows and columns, is invertible. Then, by the Implicit Function Theorem, in a neighborhood of  $p$  there exists a continuous and differentiable function  $g$  such that  $p^* = g(p^{*\perp})$ . This identifies  $p'$ , whose components are  $p^*$  and the level set of  $g$  corresponding to  $p^*$ , such that  $(H, X)(p') = 0$ . The level set defines the finite flexing of the framework. Therefore the system is not rigid.  $\square$